Some Aspects of Quasi-Quanta

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Abstract

This paper discusses $\hat{q}$-quanta, or *quasi-quanta*, which are entitities which contribute very high order terms to their environment. If the density of these $\hat{q}$ is sufficiently high, then they exert measurable effects upon nearby quanta. The paper takes a context-aware approach, following [4].

While two quanta q_i and q_j may be isomorphic, quasi-quanta may only be quasi-isomorphic. Thus, the paper is an investigation into how quasi-isomorphisms (quasi-isos or just *quasis*) can rise to the level of isomorphisms (*isos*). The thesis is that they exhibit effects in units related to the string length g_s.

Further, we also describe certain limits as *regulators* which control many downstream variables, and thus influence the measurable value associated with particles. Due to the particle-string equivalence, we are able to labor this analogy further by describing the quantum wells surrounding the quanta q as B-branes.

We make light use of topos-theoretic concepts, for the sole purpose of illustrating quantum contextuality. When we fix a context across multiple B-branes, we get a non-zero phenomenological velocity (PV). Further, we posit that both zero and infinite PV's are unphysical. Thus, we are forced to quantize the PV. However, despite the discreteness of PV, we maintain that smoothness is a sufficient parameter in recovering physicality.

Acknowledgments I'd like to thank Parker Emmerson for helpful corrections to this paper. Furthermore, the usage of the pronoun "we" in this paper is in reference to all diaspora of my folk; i.e., people who think the same as I do. This, I credit to [21], who impressed upon me the importance of diaspora. Lastly, I credit Allah for providing me with discernment to labor as I did during the production of this paper.

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Notation Throughout, let Ω be the set of truth-values assignable to a measurement μ in a context $\mathtt{C}$. Let $\mathcal{B}$ denote a brane, q a quantum, and $\hat{q}$ a "quasi-quantum," a subobject whose meaning shall arise naturally. Latin letters written in "code" font, i.e., $\mathtt{A}$, $\mathtt{B}$, etc., shall be called "contexts," as in [4]. N and k shall denote indices, but of different natures which will become apparent. Let $\lim_{a \to b}$, for arbitrary a and b, denote a *regulator* which takes in a limit and controls several other functions. Δ will represent a simplex, and δ the uncertainty of a μ.

As is usual in mathematics, $\exp(i\theta) = e^{i\theta}$ for an arbitrary θ. This functions as a "phase shift," which is particularly useful when describing the winding of a string. We do not limit ourselves to natural values of the winding number, like traditional approaches, but instead use a fuzzy version of the string winding number. This is useful, because it allows us to describe partial windings of strings around an axis.

We will let $\times$ denote the fibered (Cartesian) product of module-categories, and $\mathcal{G}^{\infty}_{\text{Glob}}$ a globular ∞-category.

The $\bowtie$ symbol, taken from [21], shall mean "turns into," or "becomes;" i.e., something of a cross of the arrow symbol and the equals symbol, informally speaking. This is an outgrowth of the Cascadian synergetics paradigm. For instance, for a map $\mathfrak{U} : U_{t_0} \to U_{t_1}$, we get $U \bowtie U$. This symbol is appealing, because it is remniscent of a lightcone; the l.h.s. being the "past" cone, and the r.h.s. being the "future" cone.

Organization The first portions of the article are devoted to explaining the core ideas, and the latter sections delve into the more "canonical" stringy aspects. We make the assumption that quasi-quanta can be identified with the *isotropy group* of monopoles, but we do not define that notion here. Instead, we focus on compactifications of (five/six)-branes.

Section 1 sets the scene for the overall paper; section 2 focuses on the picture of quantum wells in terms of ∞-groupoids, and defines limits as regulators. In section 3, we describe homology-type aspects of the $\hat{q}$-quanta, and conclude with a notion of quantum causation (alias: *quasation*).

Finally, section 4 deals with ghost terms, monopoles, and compactifications. This is by far the most incomplete section, because it relies on a rather obscure version of F-terms which is not fully elucideated here.

1 Prolegomenon

One of the most remarkable aspects of modern physics is the cut-throat level of abstraction which allows for cross-pollenation with the bleeding edge of mathematics. Many concepts of "pure mathematics" turn out to play a crucial rôle in domains such as string theory (ST), topological quantum field theory (TQFT), and condensed matter physics.[1] For example, B-branes admit a very nice description in terms of the category $\mathcal{D}^b\mathrm{Coh}(X)$ of derived, coherent bounded sheaves[2]. Another example comes from Frobenius algebras, which can be used to construct string worldsheets in $(2+1)$-dimensional TQFT [18]. The list seems virtually inexhaustible at this point, and it is hard to envision that when quantum mechanics was first developed in the early 20th century that its practitioners would imagine such advancements. However, one of the most surprising renovations of quantum theory comes from logic. Work by authors such as Isham and Butterfield [16], and Döring and Frembs [7] has clarified that we must take seriously the idea of quantum contextuality. As an extension of the observer effect, many subtleties make the quantum realm highly sensitive to measurements.

For the sake of simplicity, let $\Omega = \{\bot, \top\}$ be the set of allowable truth values of a proposition $\mathbf{p}$. Then the *measurement*

$$\mu : \mathbf{p} \to \Omega$$

makes no sense without first defining a context. To see why this is the case, let $\mathbf{p}$ be the proposition "the string is in a state of maximum tension." Obviously this may be true at one time $t = 0$ and false at another time $t' \neq 0$. So, we can define a context $\mathbf{T} \coloneqq [t]$ which encodes all of the possible values of time; then, we rewrite the measurement as

$$\mu_{\mathbf{T}} : \mathbf{p}_{\mathbf{T}} \longrightarrow \Omega_{\mathbf{T}}$$

and for a time-step $t \mapsto t'$, our measurement may either change or remain constant. If we fail to specify an appropriate context for our measurement, then we have embedded our hidden assumptions into the measurement, and thus we lose some clarity. While the example given here is very simple, it does not take much work to extend it to rather elaborate scenarios. For instance, instead of thinking merely of strings or particles, we can envision a debate between two minds $\mathscr{A}$ and $\mathscr{B}$ which attach varying epistemic weights to each parent of the proposition. The proposition is viewed as a child of weighted belief functions, and the set of truth-values may also vary from agent to agent.

Returning to the domain of physics, a classic predicament, known as "Heisenberg's uncertainty principle"[3] states that it is impossible to know both the position and momentum of a particle with absolute certainty. Thus, we may let the truth values be an ansatz for a fixed numerical value, assumed to be the outcome of a measurement. The set may be smooth or discrete[4], or as large

[1] For instance, the Barry phase.
[2] In the Landau-Ginzburg model; see [19]
[3] The original formulation dates back to [15], but for a modern treatment, see [13]
[4] I.e., infinite or finite

or as small as we desire. Then, we define two contexts $\mathtt{P}$ and $\mathtt{M}$ which encode the position and momentum, respectively. If $\mu_{\mathtt{P}} = k_0$ for a known truth-value k, then $\mu_{\mathtt{M}} = k_1 - \delta k_2$, where

- k_0 is a truth value in $\Omega_{\mathtt{P}}$
- k_1 and k_2 are truth values in $\Omega_{\mathtt{M}}$
- $\delta > 0$ is the uncertainty of the measurement

In principle, we can only assign arbitrary values to k_1 and k_2, meaning that there is not only a single layer of uncertainty, but at least two. Assuming δ is an arbitrary N-ary function of $\pm k_0$, then there are $N + 2$ layers of uncertainty, and in principle there could be infinite layers. What this reveals is that some contexts mutually exclude one another, and so a choice must be made whether one prefers partial certainty in many contexts or absolute certainty in just one. Yet, if only a single context can be queried, then we are essentially obtaining only a naïve measurement; thus, the paradox is that we would actually not know k_0 fully either.

For the sake of argument, let's suspend this paradox and imagine k_0 as the position of a quantum q. Since all quanta of the same kind are indistinguishable, then we have an isomorphism

$$q_{\mathtt{K}} \cong q'_{\mathtt{K}}$$

induced by an equivalence of allowable energy levels. Traditionally, quanta have been supposed to be irreducible, primitive entities; later, quantum field theory re-envisioned them as excitations of underlying fields. This lead to the idea of "virtual excitations[5]," in which quasi-physical potentials in the surrounding energy well could lead to fluctuactions of the well's own energy. In principle, if there are enough of these potentials clustered close enough together, they should be able to exert an effect on the quantum, either by modifying its position or its momentum. This tells us that at least some measurement contexts are effected by Planckian or sub-Planckian virtual excitations, which on their own are insufficient to excite a quantized string.

Thus, the notion of a quasi-quantum is born. Denote a single quasi-quantum by $\hat{q}$. Then, for N quasi-quanta packaged together with a sufficient density, there should be the generation of a single proper quantum q. Thus, we can write:

$$q = \{\hat{q}\}_{i=0}^{N}$$

where the density is controlled by a pre-specified volume form on the surrounding space. To spell this out explicitly, we can write it as:

$$q(N, \mathrm{Vol}(S)) =: q \qquad\qquad = \frac{N \cdot \mu_{\mathtt{P}}(\hat{q})}{\mathrm{Vol}(S)} \tag{1.1}$$

where the numerator is in units $g_s < \ell_P$ and $\mathrm{Vol}(S)$ is in units $\ell_P^{\dim S}$. We should be able to recover the value of $\hat{q}$ from

$$\hat{q} \;\; = \lim_{\mathrm{Vol}(S)\to 0} \quad q$$

which should obey

$$\hat{q} = \lim_{\dim S \to 0}$$

[5]See [12] for a comprehensive, but not original, perspective.

and give us the energy of a maximally compactified string. For all intents and purposes, this $\hat{q}$ behaves like a monopole, but it exerts vanishingly small (higher-order terms) contributions to the effective flux of q. However, for a set $\{\hat{q}\}_{i=0}^{N}$ of quasi-quanta, we do not measure zero energy, but rather a subatomic wave-packet travelling along a transverse string.[6] In the rest of this paper, I will investigate these quasi-quantum phenomenae, which were originally introduced by Emmerson in his "quasi-quanta language package."

2 Spatial Components of the Quantum Well

It is well-known folklore in the category-theory community that every ∞-groupoid is essentially a topological space. An ∞-groupoid is a generalization of the mathematical concept of a group. When physicists first got ahold of groups, they used them to investigate symmetries. Homomorphisms of groups can be thought of as physical symmetries, and group actions would be the different types of transformations. However, naïvely, groups have not even a possible way of encoding the contexts we have talked about here. Luckily, there is a specific type of category calld a groupoid that facilitates context-aware modeling.

2.1 Definitions

Definition 2.1. *A groupoid $\mathcal{G}$ is an object of the category* **Gpd***, with formal objects and morphisms between them, such that every morphism $f : o_i \to o_j$ has an inverse $f^{op} : o_i \leftarrow o_j$.*

Now, the following will be indespensible:

Definition 2.2. *A globular category $\mathcal{C}_{Glob}$ is a category $\mathcal{C}$ such that every object is a 0-cell, every arrow is a 1-cell, arrows between arrows are 2-cells, and so on, such that a morphsim between k-cells is a $(k+1)$-cell.*

Where this becomes important in our program, is that TQFT strings roughly correspond to path between points

$$\gamma : *_1 \to *_2$$

in a topological space; then, a sheet connecting the strings can be written

$$\gamma^2 : \gamma_1 \to \gamma_2$$

and a hypersurface connecting them would be

$$\gamma^3 : \gamma_1^2 \to \gamma_2^2$$

and so on and so forth. In an ∞-groupoid, we are allowed to extend this to an arbitrary γ^k for $k \in \mathbb{N}_0$, and for $k = 0$, we get just a single point. This point can either be represented by a $\{\hat{q}\}_{i=0}^{k}$-term or a q-term, since by a previous equation they are isomorphic.[7]

[6] Transversality is a complex topic in symplectic geometry; here, I am using the concept of a transverse wave-packet synonymously with a vibrating string cross-section. The cross-section consists of both the string γ itself, as well as a pair of manifolds $T_{\hat{q}_i}(\gamma)$ and $T^*_{\hat{q}_i}(\gamma)$ called the *tangent* and *cotangent* manifolds, respectively. These manifolds form a stringy interstice; however, as proud as I am of this theory, computationally defining interstitial fields remains an open problem.

[7] In plain English, the point may either be encoded as a composite flux of quasi-quanta, or a single quantum excitation.

2.2 Globular ∞-groupoids and categorical equivalences of branes to simplices

Let us work now with globular ∞-groupoids. Our objects are groupoids of the form $\mathcal{G}^{\infty}_{\text{Glob}}$, and morphisms are groupoid homomorphisms. The objects of the groupoids themselves are encoded as sub-objects of $\mathcal{G}^{\infty}_{\text{Glob}}$. Because they are sub-objects of a 0-cell, they are technically (-1)-cells. This means that they inherit their universal property from the structure of the endomorphism class $\text{End}(\mathcal{G}^{\infty}_{\text{Glob}})$, so that every arrow between objects $f : g_0 \to g_1$ lifts to a self-map $f^{\sharp} : \mathcal{G}^{\infty}_{\text{Glob}} \to \mathcal{G}^{\infty}_{\text{Glob}}$ which obeys the rule $(f^{\sharp})^{\flat} = f$ as in [1]. We get the following diagram:

$$\begin{array}{ccc} g_0 & \xrightarrow{\quad f \quad} & g_1 \\ \downarrow{\scriptstyle \iota_0} & & {\scriptstyle \iota_1}\downarrow \\ \mathcal{G}^{\infty}_{\text{Glob}} & \xrightarrow{\quad f^{\sharp} \quad} & \mathcal{G}^{\infty}_{\text{Glob}} \end{array}$$

where the ι are the standard inclusion maps, and there is an implied vertical transformation $\sharp : f \to f^{\sharp}$, which is natural, and a horizontal transformation $\varphi : \iota_0 \to \iota_1$, also natural, which results by pushing the first inclusion forward along one of the regular morphisms. Since f is invertible, then φ actually induces a relation $\iota_0 \simeq \iota_1$ via the injectivity of the maps ι_0 and ι_1. This relation is not an isomorphism, but a *quasi-isomorphism.*

We also have an equivalence $\mathcal{G}^{\infty}_{\text{Glob}} \cong \{*\}^{k}_{i=0}$, meaning that these globular groupoids are in fact algbraic varieties. By a substitution[8] $\{*\}^{k}_{i=0} \bowtie \{\hat{q}\}^{k}_{i=0}$, and assuming transitivity of the $\bowtie$ relation, we get

$$\{*\}^{k}_{i=0} = q$$

meaning that the quasi-quanta are the orbits of the groupoid category, or in other words our g-subobjects. This means that our quanta are manifestations of the following hom-set:

$$q \cong \text{Hom}_{\mathcal{G}^{\infty}_{\text{Glob}}}(\hat{q}_{\alpha}, \hat{q}_{\beta})$$

where $\hat{q}_{\alpha}$ and $\hat{q}_{\beta}$ are two quasi-isomorphic quasi-quanta obeying:

$$\lim_{r \to 0^{-}} \oint_{\hat{q}_{\alpha}}^{\hat{q}_{\beta}} f(\mathbf{q}^{r}) \ dq = \int_{S} f(\{\hat{q}\}^{k-1}_{i=0}) \ dq \tag{2.1}$$

assuming there is a more general equality

$$\lim_{r \to x} q^{r} = \{\hat{q}\}^{(k-1)+x}_{i=0} \tag{2.2}$$

where k is the number of quasi-quanta in the space $S \cong |G^{\infty}_{\text{Glob}}|$, where the right-hand-side is the topological realization of the globular ∞-groupoid. This formula is derived from the synergetic fact that k marked points forms a $(k-1)$-simplex. Since we can think of quasi-quanta as point-like objects, and the quantum q as an r-dimensional brane-like object, then the limit $r \to x$ gives us an equality between x-branes $\mathcal{B}^{x}_{\mathtt{F}} \cong q^{r \to x}$ and simplices $\{\hat{q}\}^{(k-1)+x}_{i=0} \cong \Delta^{k-1}$. This gives us

$$\mathcal{B}^{x}_{\mathtt{F}} \simeq \Delta^{k-1}$$

where the context $\mathtt{F}$ encodes the field, or other words, the type of force carried by the quantum.[9]

[8] The $\bowtie$ notation is taken from Kirby Urner [21], and more broadly from the synergetics community.

[9] As a good rule of thumb, when passing between more than one isomorphism, the equivalence relation generally weakens; because the field is not specified over $\Delta^{\bullet}$, we only get a quasi.

2.3 $\lim_{r\to x}$ as a regulator, as opposed to $\lim_{x\to r}$, which would lead to infinite phenomenological velocity (PV)

The parameter $\lim_{r\to x} r$ then seems to us to be very important, but what precisely does it control? I propose that it is a *quantization parameter*, which determines the precise way in which a system of virtual excitations can combine to excite a single string. I propose the following constraint:

$$\mu_{\mathfrak{P}}(\hat{q}) \propto q \tag{2.3}$$

which means that $q = A \cdot \left(\mu_{\mathfrak{P}}(\hat{q})\right)$; then, plugging in $\lim_{r\to x} r$ for A, by equation (1.1) this gives us an equality

$$\lim_{r\to x} r \;=\; \frac{N=k}{\mathrm{Vol}(S)} \tag{2.4}$$

This is actually quite surprising, since (1.1) is written in units $g_s \cdot (\ell_P^{\dim S})^{-1}$, which means that our regulator $\lim_{r\to x} r$ is not actually a dimensionless number, but is in units $g_s \cdot \mu_{\mathfrak{P}}^{-1} \cdot \ell_P^{-\dim S}$. This means that $q^{r\to x}$ has units

$$\left(g_s \cdot \mu_{\mathfrak{P}}^{-1} \cdot \ell_P^{-\dim S}\right) \cdot \left(g_s \cdot (\ell_P^{\dim S})^{-1}\right) = g_s^2 \cdot \ell_P^{\dim^2 S} \cdot \mu_{\mathfrak{P}}^{-1} \tag{2.5}$$

and by (2.2), so does $\{\hat{q}\}_{i=0}^{(k-1)+x}$. This means:

$$\mu_{\mathfrak{P}}(q^{r\to x}) = \mu_{\mathfrak{P}}(\{\hat{q}\}_0^{k-1+x}) = y\big(g_s^2 \cdot \ell_P^{\dim^2 S}\big) = yq^2 \tag{2.6}$$

where y is assumed to be nonzero by Emmerson's principle. Setting $q^{r\to x} = q$ gives us

$$\mu_{\mathfrak{P}} = yq$$

$$r = x = 1$$

and by setting the phenomenological velocity (PV) to

$$v_{\mathrm{phnm}} = \lim_{x\to r}(x-r)c$$

this gives us

$$\lim_{x\to r} c = \infty$$

meaning

$$\lim_{x\to r} = \frac{\infty}{c} = \infty$$

which is unphysical by [4].[10] To show this, consider the contrary: suppose that $\lim_{x\to r}$ is physical. Then, the time-step evolution acts as $\mathfrak{U} : (t=0) \bowtie (t=0)$. This means that time is frozen. By analogy with thermodynamics, consider that lower temperatures condense molecular bodies into states of higher density, while higher temperatures disperse them. Then, under $\lim_{x\to r}$, we get a state summarized best by a movie title: "everything everywhere all at once." The upshot is that infinite phenomenological velocity is inherently unphysical; i.e., not effectively realizable.

[10] In relativistic QFT, microcausality enforces front velocity $\leq c$; superluminal phase/group/tunneling velocities do not permit superluminal signaling. Even in a liberal paradigm, it seems reasonable to suggest that, even allowing for tachyons, $PV = \infty$ is unphysical. Thus, any way you have it, this is just not feasable.

2.3.1 Contextuality

However, consider this: suppose we have two contexts `C` and `D`. There is no pre-existing physical law that would stop us from allowing

$$\texttt{C-}\lim_{x\to r} c = \infty$$

for the first context, but

$$\texttt{D-}\lim_{x\to r} c = c$$

for the second.

2.3.2 Physical Interpretations of Contextuality

The above tells us that in one context, a quantity can be completely unphysical, but in another, it could be construed physically. The implications of this are profound, given that contexts were originally thought to be a purely logical construct. Thus, our choice of context can (at least appear to) influence the physical reality of our measurements. Therefore, we shall find it fruitful to continue this investigation.

3 Quasation: Quantum Causation

Let $\{\hat{q}\}_{i=0}^{N}$ be a quantum q_0^N. Then, let q_N^{2N} be a second quantum of the form $\{\hat{q}\}_{i=N}^{2N}$. In general, for a quantum, we shall from now on set

$$\{\hat{q}\}_{i=(k-1)N}^{kN} = q_{k-1}^{k} \tag{3.1}$$

which for $k = 2$ allows us to write q_N^{2N} as q_1^2, for q_{2N}^{3N} allows us to write q_2^3, and so on and so forth.

Now, let's proceed to define a quantum well, Q. We want our Q to at once be a space, and on the other hand a physical/dynamical system. So, in other words, it must be a fibered product of the form

$$\Pi : S \times \mathcal{G}_{\text{Glob}}^{\infty} \longrightarrow \mathscr{O}_S$$

where $\mathscr{O}_S$ are the observables of the system S. For every section $\pi_x \in \Pi$, we want a Whitney sum

$$(S \oplus \mathscr{G}_{\text{Glob}}^{\infty})|_x \coloneqq (s_x \times g_x) \longrightarrow \mathscr{O}_S|_x$$

where x is a value assigned to the basepoint.[11]

We could very well define the entire universe to be a single quantum well, but for most purposes, this is not very illustrative. This would look like:

$$Q = \bigoplus_{x=0}^{r} q_x^{x+1} \tag{3.2}$$

For the $x = 0$ case, we have a sign flip:

$$\{\hat{q}\}_{-N}^{N}$$

[11] The terminology "basepoint" is a play-on-words; it comes both from the *base and total space* namespace, as well as the *based space* constructions of topology.

and if we take the limit $\lim_{x \to r} Q$, then we get a quantum well Q_∞. In Fristonion terms, let $\mathcal{M}_i$ be a single Markov blanket; then

$$Q_\infty = \bigcup_i^\infty \mathcal{M}_i$$

and we get the entire space of possible Markov blankets in the limit $t \bowtie \infty$.

However, this is very vague and does not give us a lot of concrete ways of manipulating the data. Assuming that there is a relationship between the number of quanta and the number of Markov blankets in the universe, then this corresponds to the limit

$$\lim_{r \to \infty} q_r^{r+1} : \mathscr{O}_{Q_\infty}$$

However, we can assign cut-offs by replacing the ∞ term by a finite numerical value x, and instead this becomes

$$\lim_{r \to x} q_r^{r+1} : \mathscr{O}_S|_{x^{\dim S}} \tag{3.3}$$

and the number of isomorphisms becomes finite. Since each isomorphism is completed by a finite number of quasi-isos, then there will again be a finite number of quasi-isomorphisms. The total number of quasis is

$$\lim_{k \to x} kN \tag{3.4}$$

and taking $k = r$, then by (2.4) we get

$$\frac{N = k}{\mathrm{Vol}(S)} N \tag{3.5}$$

which becomes

$$\frac{(N = k)^2}{\mathrm{Vol(S)}} \tag{3.6}$$

which gives us an answer in units $(N = k)^2 \cdot (\ell_P^{-\dim S})$.

3.1 Transitions between $\hat{q}$-quanta

Suppose we are given a simplex Δ^{k-1}. This alone tells us nothing about the fields within the quantum well $Q = \Delta^{k-1} \times \mathcal{G}^\infty_{\mathrm{Glob}}$. However, if we are given a family of sections $\{\pi\}_{x \in X_{\mathtt{F}_i \in [F]}}$ for a class X of permitted values, then we can begin to reconstruct some of the $\mathtt{F}_i$ if we have an idea about which fields belong to the class $[\mathtt{F}]$. For instance, suppose we observe that all the π_x belong to an algebraic variety V. Then, the generating equation of V is part of the moduli space of generators for $\Delta^{k-1}_{\mathtt{F}_i}$. We assume

$$\mathtt{F}_i : [\mathtt{F}] \subseteq \mathcal{M}od(V)$$

where the right-hand term is the moduli space of V's.

Then, the generators are $(k-1) \times \max(i)$ matrices. Since it is easiest to compute the diagonals of square matrices, we will fix $\max(i) = k - 1$.

3.2 3-$\hat{q}$ quantum

The simplest case is that with three quasi-quanta. It is a two-dimensional scenario, and the generators become the canonical *Pauli matrices*:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and taking $\sigma_\bullet^T$ reverses the parity of the quantum. The quantum itself becomes a *Čech nerve*-type construction on three copies of the brane $\mathcal{B}^2_{\mathtt{F}_i}$. Writing this down gives[12]

$$\mathrm{Nerve}_{\check{C}}(q) \cong \hat{q}_0 \times_{\mathcal{B}^2_{\mathtt{F}_i}} \times \hat{q}_1 \times_{\mathcal{B}^2_{\mathtt{F}_j}} \hat{q}_2 \tag{3.7}$$

which gives a six-brane $\mathcal{B}^6_{\mathtt{F}_{ij}} = Nerve^3_{\check{C}}(q)$.

As an ansatz, let's suppose the determinants of the Pauli matrices determine the orientation of the sixbrane. We get

$$\mathrm{Or}(\mathcal{B}^6_{\mathtt{F}_{ij}}) = \det(\sigma_x) \times \det(\sigma_y) \times \det(\sigma_z) = -1^3 \tag{3.8}$$

and supposing the orientation also be proportional to a function of the fermion count, we have:

$$\mathfrak{F}\Big((-1)^F\Big) \propto \mathrm{Or}(\mathcal{B}^6_{\mathtt{F}_{ij}}) \tag{3.9}$$

which means that $\mathfrak{F}$ is an involution.

Proof. An involution is an operator G such that $G^2 o = \mathrm{Id}_o$ for an object o. In this case, our object is $(-1) \in \mathbb{N}\mathbf{at}$. Since we have

$$\mathfrak{F}\Big((-1)^F\Big) \propto \mathrm{Or}(\mathcal{B}^6_{\mathtt{F}_{ij}})$$

this means that F must be an odd number by the fact that the orientation is (-1). This means we have

$$\mathfrak{F}\Big((-1)^F\Big) \propto \lambda\bigg(\mathfrak{F}\Big((-1)^F\Big)\bigg)$$

This constrains us to two possible: either $\mathfrak{F}$ is the identity operator, or it acts as $(-1)^F \mapsto \big((-1)^F\big)^F$. Since F is odd, then the right-hand side is again odd. Thus,

$$\mathfrak{F}\Big((-1)^F\Big) = \mathfrak{F}^2\Big((-1)^F\Big)$$

which proves our proposition. □

[12] Technically, we are not working with branes proper, but brane *worldvolumes*, which is ideally suited for a $(3+1)$-dimensional (alias: *Minkoski*) framework.

The conclusion here is that, if and only if the Pauli matrices determine the orientation of the six-brane, then any function of the fermion count determining the six-brane must be an involution. However, we should not expect this to hold true across multiple contexts. For instance, in a context $\mathtt{C}$, we may have $(-1)^{F_{\mathtt{C}}} = -1$, and for another we may have $(-1)^{F_{\mathtt{D}}} = 1$. This means that

$$\mathcal{B}^6_{\mathtt{F}_{ij};\mathtt{C}} \simeq \mathcal{B}^6_{\mathtt{F}_{ij};\mathtt{D}}$$

becomes only a quasi-isomorphism rather than an on-the-nose iso. Thus, there may be some "invisible" effects which dramatically alter the string dynamics.

If equation (3.8) is non-commutative, then the Clifford equation $\mathtt{F}_{ij} = -\mathtt{F}_{ji}$ applies. This means that when we change between these two contexts, the left-movers become right-movers and vice-versa. Then, attempting to take the topological quotient of one by the other will result in the singleton space $\mathrm{Quot}_{\mathcal{B}^6_{\mathtt{F}_{ij}}} = \{-1\}$, which is nothing other than a Kaluza-Klein monopole with the toroidal compactification. We will discuss this further in a later section, since a lot is known about these monopoles.[13]

3.3 Changing Scenarios With Constant Context

Suppose we have two six-branes and one measurement scenario $\mathcal{S}_1$ for the first and another $\mathcal{S}_2$ for the second. Suppose we have a fixed context $\mathtt{C}$. Then, the map $\mathcal{S}_{12} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ preserves the measurement context, so that if we measure a single proposition p_t in the first, then at a later time $p_{t'}$ we measure it again and get a map $\mu_{\mathtt{C}}(\mathcal{S}_{12}) \coloneqq \mu_{\mathtt{C}}(\mathbf{p}_t) \to \mu_{\mathtt{C}}(\mathbf{p}_{t'})$. Now, let the proposition be about a string γ. Since we the scenarios differ do to taking place across different branes, then if $\mu_{\mathtt{C}}(\mathcal{S}_{12})$ is constant, meaning we obtain the same value of a proposition like "γ has an endpoint on this brane," this means the string must have travelled from the one location to the other. Thus, there is a non-trivial velocity when pushing forward along the transition $\mathcal{S}_{12}$. However, for *negative propisitions*, i.e., of the form $\neg p$, there is potentially infinite uncertainty in our measurement, since we are attempting to measure the non-existence of a truth-value in the *complement* of the brane, which amounts to trying to prove a global statement for Q^∞. The only such statements we can prove of this form are related to the identity of the string itself; i.e., $p = \gamma \neq Q^\infty$, which is obviously true across both of the branes. This is tantamount to saying that a finite algebraic variety $\mathcal{V}$ of the same dimension as the system S does not equate to the infinite moduli space $\mathcal{M}od_\infty(\mathcal{V})$.[14]

3.4 Quasi-Isomorphisms and Quasation

Since $\hat{q}$-quanta can only be *quasi-isomorphic* to one another, then if we define a "causal isomorphism" somehow, then we could not force it to act between them no matter how strongly correlated they are. This is because they contribute terms of an order too high to be causally effective. However, since by an earlier stipulation we constructed simplices from them, they could in principle play a meta-causal role. In lower dimensions, a single quasi-quantum exerts a relatively greater influence than in higher dimensions; the effect of each additional $\hat{q}$ weakens when we either extend our volume

[13] They are actually truncations of the type IIA monopole action; see [2] for more info.

[14] We can take this one step further and say that a single theory is never equivalent to the space of all possible theories, or that a single wall is never equivalent to Hilbert's hotel; these both follow because every proper subset of a set is by definition not equivalent, and since we are working with finite and infinite sets, the first will always be a proper subset of the second; viz. $A \subsetneq B$. This is a consequence of Halmos' axiom of extensionality. [14]

or our dimension. However, in the case of a $3\hat{q}$ scenario, like we have illustrated above, adding a new quasi-quantum drastically alters the geometry of our brane.

By hypothesis, in the limit $\lim_{N\to\infty}\{\hat{q}\}_{i=0}^{N}$, we obtain a copy of Q^{∞} with the property that its dynamics depend only on a single q. This would be a state of maximum entropy, and by the well-accepted axiom that entropy increases over time, this would indicate a state of infinite time, which would then imply a PV of zero. By Emmerson's law, $0 \notin \mathbb{E}$, meaning zero energy states are impossible, and thus this becomes unphysical.

From now on, set

$$\begin{aligned} \varprojlim q &= \hat{q}_0 \\ &= \lim_{0\leftarrow N}\{\hat{q}\}_{i=0}^{N} \\ &= q \bowtie \hat{q}_0 \end{aligned} \tag{3.10}$$

to be the pyknotic/inverse limit of a quantum. By hypothesis, this introduces a trivial flux term $F_{(0)}$, which does not survive to the level of an observable. In other words, $\varprojlim q$ is a nullity term which causes the observable effects of a quantum to "vanish"; i.e., they are measured as a number near zero with a high uncertainty $\delta' \gg \delta^u$, where $u \in \mathbb{N}$ and the original δ lies somewhere in the interval $[\frac{1}{4}, \frac{3}{4}]$. Then, by $\gg$, in this context, we mean $\delta' > \frac{1}{2}d(\delta, (\frac{3}{4})^n)$, where $d = d^1$ is the one-dimensional Euclidean distance.

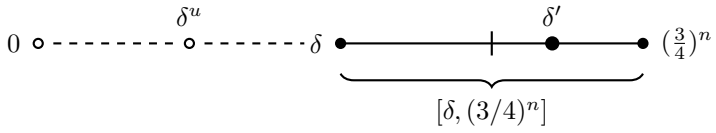

Fig. 1

The dashed lines indicate the range of uncertainties for a measurement $\mu_{\mathsf{C}}(q)$, whereas the filled lines represent the spectrum of uncertainties of the pyknotic limit $\varprojlim q$. We could have chosen any proper sub-interval of $[0, 1]$, but this choice is most convenient because powers of 2 are easier to work with for the denominator, and it also makes sense from an information-theoretic/bitwise perspective.

3.5 Quasation as a Relative Category and Homotopy of Causes

We formalize "quasation" (causation modulo smoothing gauge) as a relative category.

Definition 3.1 (Excitation category)**.** *Objects are contextual excitations* ($\mathcal{C} = [\mathsf{c}], X$) *(spectra of states, moduli spaces of operators, branes in a fixed sheaf). Morphisms are pairs* (φ, f) *with* $\varphi : \mathcal{C} \to \mathcal{D}$ *a refinement and* $f : X \to u^*Y$ *a morphism in the relevant sheaf.*

The reason why it makes sense to identify branes with spectra/moduli, rather than pure states or operators, is that the geometry of a brane is such that it serves as a sort of "micro-universe", i.e. $Q^{\infty}|_{\mathcal{B}\cong S}$ where excitations live, rather than an excitation itself, which would be more akin to a vibrational mode or state of the attached strings. Now, we introduce a *disconnected Markov*, $\mathcal{M}_{\text{disc}}$, which performs the same function as an instantiated isomorphism:

Definition 3.2 (Relative category of smoothing)**.** *Declare* (φ, f) *a weak equivalence if* f *is a smoothing equivalence (induces isomorphisms in* $\mathcal{M}_{\text{disc}}$ *after applying a regulator and passing to the discrete moduli). Let* **Ex** *be the resulting relative category* $(\text{Ob}, \text{Mor}; \mathsf{W})$*.*

We have not yet defined what an "instantiated isomorphism" is, but let us do so informally: if $\mathtt{C}$ is a class of categories, then an isomorphism $\mathtt{C} \cong \mathtt{D}$ is called *instantiated* if and only if it is realized physically. In other words, if and only if it can be written as a composition of quasi-isos of strings

$$\sum_{(i\cong j)=0}^{N} \{\gamma_i \simeq \gamma_j\}$$

where the sum is taken by counting all of the q-isomorphisms induced by the quasis. Then, it behooves us to provide just one more definition:

Proposition 3.1 (Localization)**.** *Under mild set-theoretic hypotheses (e.g., smallness conditions on hom-sets), the Dwyer–Kan simplicial localization* $L(\mathbf{Ex}, \mathsf{W})$ *exists and yields a homotopy category* $\text{Ho}(\mathbf{Ex})$ *encoding quasi-causal influence modulo smoothing gauge.*

and finally, the idea of quasation is made fully rigorous from a mathematical standpoint.

4 Treatment of Monopoles: Compactifications

As we have touched upon, a quotient space of B-branes can become a monopole in the case where the two underlying contexts of the branes differ, yet all other features remain fixed. Now, let us re-fashion some of our notation.

Let $\text{Wind}(\gamma) > 0$ be the winding number of a fundamental string γ. Then, the map

$$\text{Wind}(\gamma) \longrightarrow 0$$

can, in some cases, be induced by the presence of a Kaluza-Klein (K-K) monopole. [11] This means that, under the limit $\beta : \lim_{\dim S \to 0} \gamma^{\dim S}$, called the *Stone-Čech compactification* of the fundamental string, the string can become unwound by the pyknotic term $\varprojlim q$. Since most sources (e.g. [11], [6]) discuss K-K compactifications as compactifications of fivebranes (not sixbranes), it merits introducing a map:

$$\text{Nerve}_{\check{C}_{(1)}}(q) \longrightarrow \text{Nerve}_{\check{C}_{(2)}}(q)$$

from the fibered product of quasi-quanta surfaces to a five-brane. This generates a brane $\mathcal{B}^5_{\mathtt{F}_{ij}} \simeq \Delta^5$ which is fundamentally different from the original. In particular, one of the $\hat{q}$ is "deleted," or in other words sent to a zero-section π_0. The deleted $\hat{q}$ contributes a ghost term $F^{\text{ghost}}_{(0)}$ to the observable $\mathscr{O}_S|_{\pi_0}$, which manifest not in the measurement itself, but in the *modified uncertainty* δ'.

4.1 Phase-Shifts of Chan-Paton Factors

Let Λ be a Chan-Paton factor (CPF). Then, the unwinding of the fundamental string γ induced by the KK monopole induces a further transformation:

$$\mathrm{Tw}^{\mathrm{op}}(\gamma) = e^{i\int_{S^3} q} + e^{i\int_{T^5} q'} \tag{4.1}$$

which undoes the twist $\mathrm{Tw}(\gamma) = e^{i\Phi_0}\gamma$ induced by the family of fields $\Phi_0 = \Lambda_{\mathtt{F}_0}$. Here, we are assuming q and q' are distinct quanta. Since distinct quanta are written q_k^{k+1} and q_l^{l+1} with $l \neq k$, this means that there is a proportionality factor $\xi \in \mathbb{Q}$, which is a rational number so that $l = \xi k$. This gives us a re-write:

$$\begin{aligned}\mathrm{Tw}^{\mathrm{op}}(\gamma) &= \left(\exp(i\int_{S^3} q_k^{k+1}) + \exp(i\int_{T^5} q_{\xi k}^{\xi k+1}) \right)[\gamma] \\ &= \left(\exp(i\Lambda_{\mathtt{F}_0})\gamma \right)^{\mathrm{op}} \\ &= \left(\cos\Lambda_{\mathtt{F}_0} + i\sin\Lambda_{\mathtt{F}_0} \right) \times \gamma\end{aligned} \tag{4.2}$$

In the equation (4.1), S^3 denotes the 3-sphere, and T^5 denotes the 5-torus, as in [11], which represents the spatial limit $\pi_{x^5} \to \infty$, which corresponds to sending an observable in the 5-d axis to Q^∞. Since we originally worked with sixbranes, this leaves us with a conserved extra dimension $[\mathtt{T}]$ which controls the smoothed maps $t_0 \to t_{0+\varepsilon}$ where ε is arbitrarily small but positive.

4.2 Ghost Terms

The monopole $\varprojlim q$ picks up a Ramond-Ramond term

$$B_{\mu\nu} \cong (1+\xi)F_{(0)}^{\mathrm{ghost}}$$

from the compactication of the 10-dimensional string metric. On the other hand, it picks up a term

$$H_{\mu\nu} \cong \left((-1)^F\right)^{\mathrm{ghost}}$$

from the H-sector, which is a Kalb-Ramond contribution contributed from the context in which $\varprojlim q$ is an H-monopole. Since these are both "ghost" terms, they do not carry an effective contribution to q; in other words, they do not modify the $\mathbb{Z}_2$-graded torsion of the quantum (with respect to the fivebrane), but instead they contribute a $\mathbb{Z}_N$ term for large supersymmetry number N.

Let $\mathbf{Terms}_{[\mathtt{F}]}$ be the category whose objects are terms of the form $F_{\mu\nu}$ with L an arbitrary field. Then, $\mathrm{End}(\mathtt{F}) = \underline{\mathrm{Hom}}_{[\mathtt{F}]}(\mathbf{Terms}_{[\mathtt{F}]}, \mathbf{Terms}_{[\mathtt{F}]})$. The morphisms are homomorphisms of groups; for instance, the $U(1)$ charge of the monopole, or the $SU(2)$ charge of a multiplet. In this category, there is a sequence:

$$(1+F_{(0)})\cdot\hat{q} \to B_{\mu\nu} \xrightarrow{\mathbb{Z}_2} H_{\mu\nu} \xrightarrow{\mathbb{Z}_{2\to k}} Q_{\mu\nu} \xrightarrow{\varprojlim q} (1+F_{(1)})\cdot\hat{q} \tag{4.3}$$

where the generic quasi-quantum is a 0-object; i.e., both the *ind-object* and the *pro-object* of the sequence. On the far-left side, the term $(1+F_{(0)})$ induces a quasi-isomorphism $\hat{q} \simeq \hat{q}'$ for all $\hat{q}$, but on the far-right side, the term $(1+F_{(1)})$ induces an isomorphism $\left(f^\sharp(\hat{q}) = q\right) \cong \left(f^\sharp(\hat{q}') = q'\right)$, where $f^\sharp$ is the enrichment/promotion functor [1].

Smoothing Profiles The limit $\mathcal{R} : \lim_{k\to\infty}(1+F_{k-1})$ shall be called a *mollifier* or *smoothing profile* according to the following definition:

Definition 4.1. *A smoothing profile (mollifier) $\mathcal{R}$ is a functor promoting a k-category to an ∞-category, such that pushforward along $\mathcal{R}$ results in a self-map of Q^{∞}.*

Remark 4.1. *By ∞-category, we mean a quasi-category, as in the sense of [20].*

Definition 4.2. *The image/pushforward of a mollifier $\mathcal{R}(\mathcal{C})$ shall be called a* smoothing *of $\mathcal{C}$.*

Axiom 4.1. *Smoothing is a* sufficient*, but not necessary condition for physicality in the context of gravitation.*

The thesis that we have been operating under throughout this work is that, for sufficiently high N, the function $\mathscr{F} \coloneqq f^{\sharp^N}(\hat{q})$ actually becomes *effective*. Letting $\Omega = \mathbb{N}_0$, this means that we have $\mu_{\mathcal{C}}(\mathscr{F}(\hat{q}))$ is mapped to an integer ≥ 0. This means that the functor $\mathscr{F}$ gives a sort of "torsion," considering that by [17], torsion is always integer-valued. However, we should not think of $\mathscr{F}$ *purely* as torsion, but instead it should be interpreted as the *application of torsion* to a quasi-quantum. If we take this idea seriously, then we are left to conclude that quanta are $\hat{q}$-torsors. However, they are not singular torsors in the sense that they preserve just a single $\hat{q}$. but they are *family torsors*, meaning they preserve the family $\{\hat{q}\}_{i=0}^{N}$.

By [2], the K-K monopole is T-dual to the solitonic five-brane. The upshot is, the compactification Stone-Čech β is a manifestation of T-duality. The caveat is that the original authors only considered the M-theoretical case of an 11-dimensional compactification. This does not align with the presentation we have given thus far. Thus, we have an ambiguity: either we modify (4.1) to include more S-dimensions, or we raise the dimension of the torus T^5. Since T^5 is a bona-fide fivebrane, it suffices to raise the dimension of S via a map $S^{3\to 6}$; thus our spatial limit becomes $\mathcal{B}^6_{\bullet} \times \mathcal{B}^5_{\bullet'}$. We can understand this as two copies of $\mathcal{B}^5_{[\mathbf{F}]}$, but with the addition of a forward time-dimension for the first. The transition function between them then becomes a nerve-type construction:

$$\mathcal{B}^{(5+\frac{1}{2})\pm\frac{1}{2}}_{[\mathbf{F}]_{ij}} = \bigotimes_{k=1}^{\frac{11\pm 1}{2}} \mathcal{B}^k$$

where the $\pm$ term is given by the probability of transitioning, and in the case where the operator $\pm 1 = +1$, then we are working in the critical dimension of M-theory, and the source of the compactification is none other than a Calabi-Yau manifold. We will not discuss these dynamics here. The case we have worked with thus far is the $\pm 1 = -1$ case; i.e., the case of $(-1)^F$ for odd F. We could very well work in the M-theory picture, although some adjustments must be made.[15]

4.3 Reciprocal Compactifications and Non-Relativistic Limits

Recent work by Lambert and Smith [3] on non-relativistic decoupling limits of string and M-theory introduces a reciprocity between dual Newton–Cartan (NC) compactifications. Given a pair of intersecting branes (p, q) that admit a quarter-BPS configuration, one may take either a p-brane

[15] We will not make these adjustments here, sadly; that is a story for another day. Such a formulation would have strong implications for supergravity; see [5] and [3] for the relevant details.

NC limit of the q-brane worldvolume theory, or the reciprocal q-brane NC limit of the p-brane worldvolume theory. The remarkable result is that these two limits commute:

$$\mathcal{C}_p^{(q)} \;\simeq\; \mathcal{C}_q^{(p)}, \tag{4.4}$$

implying an equivalence between the corresponding compactified sectors. In the quasi-quantum framework, this reciprocity condition provides a geometric completion of the quasi-isomorphism $\hat{q}_\alpha \simeq \hat{q}_\beta$ described in Section (2.2) Each quasi-quantum monopole can thus be viewed as a fixed-point of a reciprocal compactification pair, realizing a non-relativistic duality between brane and anti-brane sectors.

Quasi-Quanta as Self-Dual Strings. In the M-theoretic setting, the M2–M5 intersection considered in [3] yields a pair of reciprocal limits, the M2NC and M5NC compactifications. The decoupling limit isolates the self-dual strings that propagate along the intersection. Identifying each $\hat{q}$ with the endpoint of such a self-dual string on a B-brane allows us to reinterpret the quasi-quantum as a local Newton–Cartan defect within the compactified background. The moduli space metric of these self-dual strings then provides a concrete realization of the quantized phenomenological velocity (PV)

$$v_{\text{phnm}} \;\longleftrightarrow\; \lim_{c\to\infty} \frac{E_{\text{long}}}{E_{\text{trans}}}, \tag{4.5}$$

where E_{long} and E_{trans} denote the longitudinal and transverse Newton–Cartan energy densities, respectively.

Consequences.

1. *Ghost terms* appearing in Section 4.2 acquire a geometric interpretation as the residual gauge fields that preserve reciprocity under NC–Weyl transformations.
2. *Phase shifts of Chan–Paton factors* correspond to local scale transformations $c \mapsto \Lambda c$ of the Newton–Cartan background, maintaining the symmetry group of the compactified manifold.
3. The hierarchy of compactifications can thus be summarized as

$$\text{quasi-quantum monopole} \;\subset\; \text{reciprocal brane pair} \;\subset\; \text{Newton–Cartan geometry}. \tag{4.6}$$

This perspective clarifies the physical role of the $\hat{q}$: it mediates between dual compactifications, serving as a quasi-isomorphic bridge between relativistic and non-relativistic regimes. In this sense, the monopole is not merely a topological defect but a regulator of reciprocity across NC limits.

4.4 Ehresmann Interpretation of Longitudinal and Transverse Energies

Let $\pi : \mathcal{E} \to \mathcal{B}$ denote the projection from the total quasi-quantum bundle $\mathcal{E}$ onto a base manifold $\mathcal{B}$ encoding the external spacetime or worldvolume coordinates of the brane. An Ehresmann connection on π is a splitting of the tangent bundle

$$T\mathcal{E} = H \oplus V, \tag{4.7}$$

where $V = \ker d\pi$ is the *vertical bundle* and H is its horizontal complement. The connection 1-form $\omega : T\mathcal{E} \to V$ projects any tangent vector onto its vertical component, while the horizontal projection $h = \mathrm{id} - \omega$ isolates the lifted base dynamics.

Energetic decomposition. Within this framework, the two Newton–Cartan energy densities introduced in Section 4.3 can be understood as

$$E_{\text{long}} = \big\langle \mathcal{L}_{hX}\, g_{NC},\, hX \big\rangle, \tag{4.8}$$

$$E_{\text{trans}} = \big\langle \mathcal{L}_{vX}\, g_{NC},\, vX \big\rangle, \tag{4.9}$$

where $X \in T\mathcal{E}$ is a dynamical vector field generating quasi-quantum flow, and hX and vX denote its horizontal and vertical components. The metric g_{NC} is the degenerate Newton–Cartan bilinear form acting separately on H and V.

The ratio

$$v_{\text{phnm}} = \frac{E_{\text{long}}}{E_{\text{trans}}} = \frac{\|hX\|^2_{g_{NC}}}{\|vX\|^2_{g_{NC}}} \tag{4.10}$$

thus quantifies the relative curvature between the two Ehresmann subbundles. In the reciprocal compactification picture, it measures the anisotropy of energy transport between the external brane geometry and its compactified internal sector.

Physical interpretation. Horizontal lifts correspond to propagation along the brane or base manifold, governed by the *longitudinal connection curvature* $\mathcal{R}_H = d\omega_H + \omega_H \wedge \omega_H$, while vertical curvature $\mathcal{R}_V$ encodes the self-interaction of internal quasi-quanta confined to the fiber. The quasi-quantum monopole $\hat{q}$ then appears as a defect where the curvatures balance:

$$\mathcal{R}_H \simeq \mathcal{R}_V \quad \Longleftrightarrow \quad E_{\text{long}} \simeq E_{\text{trans}}, \tag{4.11}$$

signifying the onset of a self-dual or isotropic regime of the Newton–Cartan geometry. Deviations from this equality generate finite phenomenological velocities, while equality corresponds to a static, reciprocal compactification fixed-point.

Summary. In this way, the Newton–Cartan energy densities admit a purely geometric interpretation in terms of an Ehresmann connection:

$E_{\text{long}} \leftrightarrow$ horizontal lift (external dynamics), $\qquad E_{\text{trans}} \leftrightarrow$ vertical curvature (internal dynamics).

The phenomenological velocity v_{phnm} becomes a curvature ratio linking base and fiber energetics, providing a natural bridge between quasi-quantum mechanics and the differential geometry of reciprocal limits.

4.5 Abelian Regularization of Compactified Quasi-Quanta

In the quasi-quantal formulation, the compactification of a string background induces a discretization of the quasi-quanta operator

$$\hat{q} \;\sim\; n\, R^{-1}, \qquad n \in \mathbb{Z},$$

whose eigenvalues encode the quantized momenta along the compact cycle. The corresponding quasi-energy spectrum

$$E_n = \sqrt{m^2 + \hat{q}_n^2}$$

enters all compactified observables, such as the one-loop vacuum energy or modular partition function, through sums of the form

$$S(N) = \sum_{n=1}^{N} f(\hat{q}_n),$$

where $f(\hat{q}_n)$ represents the quasi-quantal contribution of the nth mode.

Since the operator $\hat{q}$ inherits the discrete topology of the compact direction, the sum over its spectrum is typically divergent in the unregularized limit. To make sense of such expressions, we introduce *Abel's summation formulae*, which recasts the discrete quasi-quantal sum into a continuous, geometrically meaningful integral:

$$\sum_{n=1}^{N} a_n b_n = A_N b_N - \int_1^N A(x)\, db(x), \qquad A(x) = \sum_{n \le x} a_n. \tag{4.12}$$

Setting $a_n = f(\hat{q}_n)$ and choosing $b_n = e^{-\epsilon n}$ with $\epsilon > 0$ as an Abel regulator, we obtain

$$\sum_{n=1}^{\infty} f(\hat{q}_n) e^{-\epsilon n} = \lim_{N\to\infty} \left[A_N e^{-\epsilon N} + \epsilon \int_1^N A(x) e^{-\epsilon x}\, dx \right].$$

The exponential regulator $e^{-\epsilon n}$ plays the role of a *tempered connection* on the discrete base of $\hat{q}$'s spectrum, ensuring convergence and allowing analytic continuation as $\epsilon \to 0^+$.

In this framework, Abel's summation isolates the divergent portion of the quasi-quantal spectrum into the boundary term $A_N e^{-\epsilon N}$, while the integral term encodes the smooth envelope of the compactified excitation structure. The vacuum energy thus becomes

$$E_{\text{vac}}(R) = \frac{1}{2} \sum_{n=-\infty}^{\infty} E(\hat{q}_n) \xrightarrow{\text{Abel}} \frac{\epsilon}{2} \int_0^{\infty} A(x) e^{-\epsilon x}\, dx,$$

providing a manifestly convergent representation of the quasi-quantal field energy.

Abel-regularization therefore reveals the geometric continuity hidden within the discrete operator $\hat{q}$: while $\hat{q}$ quantizes motion along compact fibers, Abel's formula restores the differential structure that links these quasi-quanta to the smooth base of the total space. In this sense, the Abelian sum defines a bridge between the *discrete vertical dynamics* of the compactified sector and the continuous horizontal geometry of the background connection—mirroring the dual interpretation of E_{long} and E_{trans} in the Ehresmann framework.

5 Conclusion

In this paper, we have proposed a context-aware formalism for quasi-quanta, entities that inhabit the liminal zone between virtual excitations and proper quantum degrees of freedom. Beginning from the assumption that quasi-quanta $\hat{q}$ contribute higher-order terms to their local environment, we reinterpreted these effects in terms of categorical and topological structures: globular ∞-groupoids, quasi-isomorphisms, and limits-as-regulators that encode phenomenological velocity (PV). The outcome is a unified picture in which contextuality, topology, and quantization appear as facets of the same underlying structure.

Sections 2 and 3 demonstrated that the interplay between context and limit processes allows quasi-isomorphisms to ascend to true isomorphisms under appropriate regulation. The regulator $\lim_{r\to x}$ was reinterpreted as a quantization operator that prevents the emergence of unphysical, infinite PV, ensuring that physicality remains tied to smoothness. The quasi-causal framework (*quasation*) then offered a homotopical interpretation of quantum causation as morphisms between regulated contexts.

In Section 4 we extended the formalism to include compactifications and monopole-like quasi-quanta. By incorporating the results of Lambert and Smith [3], we identified a natural correspondence between quasi-quanta and the self-dual strings arising in reciprocal non-relativistic (Newton–Cartan) limits of M-theory. The equality of reciprocal compactifications, $\mathcal{C}_p^{(q)} \simeq \mathcal{C}_q^{(p)}$, mirrors the quasi-isomorphism condition central to the quasi-quantum thesis. In this way, the quasi-quantum becomes the regulator of reciprocity between relativistic and non-relativistic regimes.

The introduction of an Ehresmann connection on the quasi-quantum bundle $\pi : \mathcal{E} \to \mathcal{B}$ provided an interpretive bridge between the geometric and energetic aspects of the theory. The decomposition $T\mathcal{E} = H \oplus V$ allowed us to define the longitudinal and transverse energy densities, E_{long} and E_{trans}, as the norms of the horizontal and vertical components of the Newton–Cartan flow. The phenomenological velocity,

$$v_{\text{phnm}} = \frac{E_{\text{long}}}{E_{\text{trans}}},$$

thus emerged as a curvature ratio measuring the anisotropy between external and internal energy flows. The equilibrium condition $E_{\text{long}} \simeq E_{\text{trans}}$ corresponds to a self-dual, reciprocal compactification—a fixed point at which the quasi-quantum becomes isotropic in its energy distribution.

Taken together, these developments suggest that the quasi-quantum is not a mere mathematical curiosity but a geometrical mediator between distinct compactification regimes. Its dynamics, when viewed through the lens of Ehresmann geometry and reciprocal limits, reveal a consistent internal logic linking categorical equivalence, energetic symmetry, and contextual physicality. The framework points toward a deeper synthesis of topological, non-relativistic, and quantum perspectives—one in which the smoothness of context and the discreteness of quantization are not contradictory, but mutually generative.

Future work will involve explicit computation of quasi-quantum moduli spaces under reciprocal compactification, and the extension of the present model to fermionic sectors and supersymmetric limits. Ultimately, the hope is that the formalism of quasi-quanta will provide a categorical bridge between the mathematical languages of higher geometry and the physical languages of field and string theory.

A.I. Disclaimer Artificial intelligence was used for the following purposes: 1.) to provide instruction on how to create Tikz diagrams; 2.) to construct the "conclusion" section (not including this paragraph), and 3.) to integrate Ehresmannian geometry with compactification frameworks. The rest of the document was written by hand and transcribed using TEXstudio on a MacBook 16" Pro, provided courtesy of Emmerson.

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